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NONLINEAR DYNAMICS OF A ROTOR WITH CANTILEVERED DISK RESTING ON ANGULAR CONTACT BALL BEARINGS

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Abstract. *The mathematical model of nonlinear oscillations of the rotor resting on angular contact ball bearings is developed. The disc is fixed on the console end of the shaft. The deflection of the shaft, and the elastic deformation of the bearings have the same order. Analysis of free oscillations is carried out, using nonlinear normal modes. The modes and backbone curves of rotor nonlinear oscillations are calculated. The system has soft characteristics.*

Key words: *rotor, ball bearing, oscillations, nonlinear normal modes, backbone curves.*

НЕЛИНЕЙНАЯ ДИНАМИКА РОТОРА С КОНСОЛЬНО ЗАКРЕПЛЁННЫМ ДИСКОМ НА РАДИАЛЬНО-УПОРНЫХ ШАРИКОПОДШИПНИКАХ

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Аннотация. *Получена математическая модель нелинейных колебаний ротора на радиально-упорных шарикоподшипниках. Диск закреплён на консольном конце вала. Прогибы вала одного порядка с упругими деформациями подшипников. Анализ свободных колебаний выполнен методом нелинейных нормальных форм. Рассчитаны формы и скелетные кривые колебаний ротора.*

Ключевые слова: *ротор, шарикоподшипник, колебания, нелинейные нормальные формы, скелетные кривые.*

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Анотація. *Отримано математичну модель нелінійних коливань ротора на радіально-упорних шарикопідшипниках. Диск закріплено на консольному кінці вала. Прогини вала одного порядку із пружними деформаціями підшипників. Аналіз вільних коливань виконано методом нелінійних нормальних форм. Розраховано форми і скелетні криві коливань.*

Ключові слова: *ротор, шарикопідшипник, коливання, нелінійні нормальні форми, скелетні криві.*

Introduction

Analysis of nonlinear dynamics of machines allows predicting destructive oscillations in conditions, which are safe from the point of view of the linear model of the system, and due to more precise definition to reduce their materials consumption and terms of design. Practically, all vehicles contain rotors supported by nonlinear bearings. Nonlinearity of ball-bearings is caused by clearances between the balls and races, and the nonlinear dependence of deformations on contact forces. Nonlinear analysis of rotors on ball-bearings with clearances are investigated in articles [1,2,3]. Closing of the radial internal clearance in ball-bearings causes shock loads and excessive vibrations. In order to reduce them, axial preload of ball bearing are used. Nonlinear dynamics of such rotors is investigated in papers [4, 5].

In the majority of works, the oscillations of rotors in which one disk is in the middle between the supports are considered, and oscillations are caused by unbalance or bearings defects. In such models, the deformed shaft center line is approximated by the harmonic functions as a rule. In many machines the rotor resting on axial preloaded angular contact ball-bearings has a disk on the console end, and the bearings are mounted on the vibrating basis. Using such a model it is possible to provide the rotors of helicopter turbines which are used as engines of dump trucks, or electric motors with impellers of pumps which are used in other vehicles.

Problem formulation

Influence of vibration of the basis on nonlinear dynamics of a rotor, which has a disk on console part of a shaft, supported by ball-bearings is insufficient investigated. Therefore the problems of creation of mathematical model and development of technique to research oscillations of a rotor on axial preloaded angular contact ball-bearings, and also the analysis of dynamics of a rotor when the frequency of its rotation is in frequency band of vibration of the basis are solved.

Equations of rotor oscillations

For the rotor of such a structure it is difficult to apply harmonic functions, therefore we use the finite element method for approximation of a deformed shaft. The design model of a rotor is presented in fig. 1. Finite elements approximate the sites of a shaft of a constant section. Disks and supports are placed in nodes. Numbers of nodal sections are denoted in fig. 1 by digits 1–5. We consider the forces and the moments of forces of inertia of a disk, and also the contact forces arising in bearings as boundary conditions in the corresponding nodes.

Free oscillations of a shaft of constant section are described by the following equations [6]

$$EI\frac{\partial^4 u_x}{\partial \zeta^4}+\rho F\frac{\partial^2 u_x}{\partial t^2}=0,$$

$$EI\frac{\partial^4 u_y}{\partial \zeta^4}+\rho F\frac{\partial^2 u_y}{\partial t^2}=0, \qquad (1)$$

where I and F – are the second moment of area and the area of the shaft, respectively, E and ρ – are Young's modulus and the mass density of the shaft, respectively.

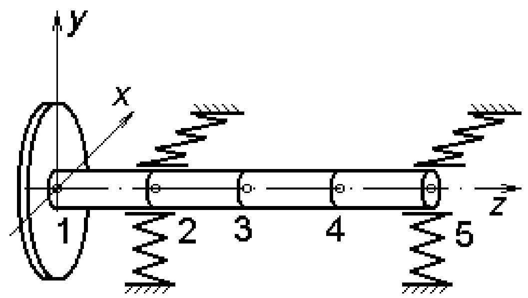

Fig. 1. Finite elements of a rotor and nodes on the ends of elements (1–5)

Coordinate axes are directed, as it is shown in Fig. 1. Generalized coordinates which are the components of a vector of nodal displacements of a node i are the following sequence $u_{i,1}=u_{i,x}$, $u_{i,2}=\theta_{i,y}$, $u_{i,3}=u_{i,y}$, $u_{i,4}=\theta_{i,x}$, $u_{i,5}=u_{i,z}$. Interpolation polynoms of finite element are the functions of a bending line of a beam at single movements of nodal sections [7]

$$N_{e,1}(\zeta)=1-3\frac{\zeta^2}{l^2}+2\frac{\zeta^3}{l^3},$$

$$N_{e,2}(\zeta)=l\left(\frac{\zeta}{l}-2\frac{\zeta^2}{l^2}+\frac{\zeta^3}{l^3}\right),$$

$$N_{e,3}(\zeta)=3\frac{\zeta^2}{l^2}-2\frac{\zeta^3}{l^3},$$

$$N_{e,4}(\zeta)=l\left(\frac{\zeta^3}{l^3}-\frac{\zeta^2}{l^2}\right), \qquad (2)$$

where l is the element length, ζ is the coordinate along an element axis. Deflections of finite element between nodes $i, i+1$ are determined by polynoms

$$u_y=N_{e,1}u_{i,1}+N_{e,2}u_{i,2}+N_{e,3}u_{i+1,1}+N_{e,4}u_{i+1,2},$$

$$u_y=N_{e,1}u_{i,3}+N_{e,2}u_{i,4}+N_{e,3}u_{i+1,3}+N_{e,4}u_{i+1,4}. \qquad (3)$$

Values $u_{i,z}$ depend only on time, because the shaft is not deformed along a rotation axis.

The equations of oscillations of a shaft are obtained by Galerkin's method at simultaneous approximation of both the equations and boundary conditions [8]

$$\int_{\mathrm{O}} W_e R_{\mathrm{O}} d\mathrm{O}+\int_{\Gamma}\overline{W}_e R_{\Gamma} d\Gamma=0, \qquad (4)$$

where $R_О$ is a residual of the solution of the equation; $R_Г$ is a residual in boundary conditions; W_e and $\overline{W}_e$ are the weight functions in the area and on border, respectively; *e* is a number of finite element. As weight functions in this method we take interpolation polynoms $W_e \equiv N_e$.

If expressions (1), (2) and (3) substituted into the first integral (4) we receive the following integrals longwise of an element

$$\int_0^l [N_e]^{\mathrm{T}} \left(EI \frac{\partial^4 u_x}{\partial \zeta^4} + \rho F \frac{\partial^2 u_x}{\partial t^2} \right) d\zeta ,$$
$$\int_0^l [N_e]^{\mathrm{T}} \left(EI \frac{\partial^4 u_y}{\partial \zeta^4} + \rho F \frac{\partial^2 u_y}{\partial t^2} \right) d\zeta . \qquad (5)$$

Carrying out in (5) integration by parts for terms, which containing derivatives on coordinate ζ, we receive

$$\int_0^l [N_e]^{\mathrm{T}} \left(EI \frac{\partial^4 u_x}{\partial \zeta^4} \right) d\zeta = = [K_e][u_{i,1} \;\; u_{i,2} \;\; u_{i+1,1} \;\; u_{i+1,2}]^{\mathrm{T}},$$
$$\int_0^l [N_e]^{\mathrm{T}} \left(EI \frac{\partial^4 u_y}{\partial \zeta^4} \right) d\zeta = = [K_e][u_{i,3} \;\; u_{i,4} \;\; u_{i+1,3} \;\; u_{i+1,4}]^{\mathrm{T}}, \qquad (6)$$

where $[K_e]$ is a stiffness matrix of finite element. Carrying out in (5) integration for terms with derivatives on time we receive

$$\int_0^l [N_e]^{\mathrm{T}} \left(\rho F \frac{\partial^2 u_x}{\partial t^2} \right) d\zeta = = [M_e][\ddot{u}_{i,1} \;\; \ddot{u}_{i,2} \;\; \ddot{u}_{i+1,1} \;\; \ddot{u}_{i+1,2}]^{\mathrm{T}},$$
$$\int_0^l [N_e]^{\mathrm{T}} \left(\rho F \frac{\partial^2 u_y}{\partial t^2} \right) d\zeta = = [M_e][\ddot{u}_{i,3} \;\; \ddot{u}_{i,4} \;\; \ddot{u}_{i+1,3} \;\; \ddot{u}_{i+1,4}]^{\mathrm{T}}, \qquad (7)$$

where $[M_e]$ is a matrix of mass of finite element. The components of lines and columns of matrixes corresponding to movement u_z will be zero because the shaft is not deformed along a rotation axis, except a diagonal component of a matrix of masses which is equal to the mass of finite element.

The first boundary condition on the end of a shaft with a disk is equality of the bending moment to the moment of forces of inertia of a disk

$$\left[EI \frac{\partial^2 u_x}{\partial \zeta^2} + I_1 \frac{\partial^3 u_x}{\partial \zeta \partial t^2} + I_0 \Omega \frac{\partial^2 u_y}{\partial \zeta \partial t} \right]_{\zeta=0} = 0 ,$$
$$\left[EI \frac{\partial^2 u_y}{\partial \zeta^2} + I_1 \frac{\partial^3 u_y}{\partial \zeta \partial t^2} - I_0 \Omega \frac{\partial^2 u_x}{\partial \zeta \partial t} \right]_{\zeta=0} = 0 , \qquad (8)$$

where I_1 and I_0 are the diametrical and polar moments of inertia of a disk, respectively, Ω is an angular speed of rotor. If expressions (2), (3), (8) and $\zeta = 0$ substituted into the second integral (4) we receive an additive to a matrix of masses $[M_{I1}]$ and a gyroscopic matrix $[G_1]$ for degrees of freedom of the corresponding node

$$[N_e]^{\mathrm{T}} I_1 \frac{\partial^3 u_x}{\partial \zeta \partial t^2}\bigg|_{\zeta=0} + [N_e]^{\mathrm{T}} I_1 \frac{\partial^3 u_y}{\partial \zeta \partial t^2}\bigg|_{\zeta=0} = = [M_{I1}][\ddot{u}_{1,1} \;\; \ddot{u}_{1,2} \;\; \ddot{u}_{1,3} \;\; \ddot{u}_{1,4}]^{\mathrm{T}}, \qquad (9)$$

$$[N_e]^{\mathrm{T}} I_0 \Omega \frac{\partial^2 u_y}{\partial \zeta \partial t}\bigg|_{\zeta=0} + [N_e]^{\mathrm{T}} I_0 \Omega \frac{\partial^2 u_x}{\partial \zeta \partial t}\bigg|_{\zeta=0} = = [G_1][\dot{u}_{1,1} \;\; \dot{u}_{1,2} \;\; \dot{u}_{1,3} \;\; \dot{u}_{1,4}]^{\mathrm{T}}. \qquad (10)$$

The second boundary condition on the end of a shaft with a disk is equality of lateral force to force of inertia of a disk

$$\left[EI \frac{\partial^3 u_x}{\partial \zeta^3} + m_0 \frac{\partial^2 u_x}{\partial t^2} \right]_{\zeta=0} = 0 ,$$
$$\left[EI \frac{\partial^3 u_y}{\partial \zeta^3} + m_0 \frac{\partial^2 u_y}{\partial t^2} \right]_{\zeta=0} = 0 , \qquad (11)$$

where m_0 is the mass of a disk. If expressions (2), (3), (11) and $\zeta = 0$ substituted into the second integral (4), we receive an additive to a matrix of masses $[M_{m1}]$

$$[N_e]^{\mathrm{T}} m_0 \frac{\partial^2 u_x}{\partial t^2}\bigg|_{\zeta=0} + [N_e]^{\mathrm{T}} m_0 \frac{\partial^3 u_y}{\partial t^2}\bigg|_{\zeta=0} = = [M_{m1}][\ddot{u}_{1,1} \;\; \ddot{u}_{1,2} \;\; \ddot{u}_{1,3} \;\; \ddot{u}_{1,4}]^{\mathrm{T}}. \qquad (12)$$

If the disk is fixed on a shaft with eccentricity *a*, then equations (11) must change so

$$\left[EI\frac{\partial^3 u_x}{\partial\zeta^3}+m_0\frac{\partial^2 u_x}{\partial t^2}\right]_{\zeta=0}-m_0 a\Omega^2\cos\Omega t=0,$$
$$\left[EI\frac{\partial^3 u_y}{\partial\zeta^3}+m_0\frac{\partial^2 u_y}{\partial t^2}\right]_{\zeta=0}-m_0 a\Omega^2\sin\Omega t=0. \qquad (13)$$

If expressions (2), (3), (13) and $\zeta=0$ substituted into the second integral (4) we receive a right-hand vector of the equations of oscillations $\{H_D(\Omega,t)\}$ which is caused by a disk unbalance besides the matrix $[M_{m1}]$

$$\{H_D(\Omega,t)\}=m_0 a\Omega^2[\cos\Omega t\quad 0\quad \sin\Omega t\quad 0]^{\mathrm{T}}. \qquad (14)$$

For the node which is fixed in the bearing, boundary conditions on axes x, y are

$$-\left(EI\frac{\partial^3 u_x}{\partial\zeta^3}\right)_{\mathrm{i}}+\left(EI\frac{\partial^3 u_x}{\partial\zeta^3}\right)_{i+1}-P_x(u_x,u_y,u_z)=0,$$
$$-\left(EI\frac{\partial^3 u_y}{\partial\zeta^3}\right)_{\mathrm{i}}+\left(EI\frac{\partial^3 u_y}{\partial\zeta^3}\right)_{i+1}-$$
$$-P_y(u_x,u_y,u_z)=0, \qquad (15)$$

where $P_{x,i}$ and $P_{y,i}$ are functions of bearing restoring forces which are received in paper [9]. If expressions (2), (3), (15) and $\zeta=0$, in case of the left node of an element, or $\zeta=l$, in case of the right node of an element, substituted into the second integral (4) we receive a vector function of the bearing restoring forces

$$[N_e]_i^{\mathrm{T}}P_{x,i}(u_x,u_y,u_z)\Big|_{\zeta=l}+$$
$$+[N_e]_i^{\mathrm{T}}P_{y,i}(u_x,u_y,u_z)\Big|_{\zeta=l}=$$
$$=[P_{x,i}(u_{i,1},u_{i,3},u_z)\quad 0\quad P_{y,i}(u_{i,1},u_{i,3},u_z)\quad 0]^{\mathrm{T}}=$$
$$=\{K_{\Pi}(U)\}. \qquad (16)$$

If the rotor is mounted on the vibrating base, the vector of kinematic excitation of oscillations is added on the right side of equation [10]

$$\{H_{\Pi}(\omega,t)\}=-[M]\{A_{\Pi}(\omega,t)\}, \qquad (17)$$

where $[M]$ is a matrix of masses, $\{A_{\Pi}(\omega,t)\}$ is a vector of vibration accelerations of support, ω is the angular frequency of vibration of support. Damping forces are concentrated in bearings. Therefore the vector of damping forces has the same structure as the vector $\{K_{\Pi}(U)\}$. In this paper we accept model of viscous damping, then coefficients of damping matrix $[C]$ placed on its diagonal in the same lines as similar component of the vector $\{K_{\Pi}(U)\}$. Assembling the matrixes which received on formulas (6), (7), (9), (10), (12) and vectors (14), (16), (17) we receive the equation of oscillations.

$$[M]\{\ddot U\}+[G]\{\dot U\}+[C]\{\dot U\}+[K]\{U\}+$$
$$+\{K_{\Pi}(U)\}=\{H_D(\Omega,t)\}+\{H_{\Pi}(\omega,t)\}. \qquad (18)$$

Analysis of free oscillations of a rotor

The equation of free oscillations without damping has a form

$$[M]\{\ddot U\}+[G]\{\dot U\}+[K]\{U\}+\{K_{\Pi}(U)\}=0. \qquad (19)$$

To analyse free oscillations, we use the method of nonlinear normal modes, which allows to transform the analysis of finite degree of freedoms system to the analysis of single degree of freedom oscillator [9, 11].

Multiply (19) by $[M]^{-1}$ and write the vector of generalized velocities $\{V\}=\{\dot U\}$ we receive the first order system of equations

$$\{\dot V\}+[G']\{V\}+[K']\{U\}+\{K'_{\Pi}(\mathrm{U})\}=0, \qquad (20)$$

where $[M]^{-1}[G]=[G']$, $[M]^{-1}[K]=[K']$, $[M]^{-1}\{K_{\Pi}(U)\}=\{K'_{\Pi}(U)\}$.

We write all phase coordinates as the functions of one pair of phase coordinates which can be chosen arbitrary [11]

$$\begin{Bmatrix}U\\V\end{Bmatrix}=\begin{Bmatrix}P(p,q)\\Q(p,q)\end{Bmatrix}. \qquad (21)$$

where p is displacement and $q=\dot p$ is velocity. We can express the components of vector functions $\{P(p,q)\}=[p_1,...,p_{\mathrm{N}}]^{\mathrm{T}}$ and $\{Q(p,q)\}=[q_1,...,q_{\mathrm{N}}]^{\mathrm{T}}$ in the form of a Taylor series:

$$p_n(p,q)=\vartheta_{n,1}p+\vartheta_{n,2}q+\vartheta_{n,3}p^2+ \\ +\vartheta_{n,4}pq+\vartheta_{n,5}q^2+\vartheta_{n,6}p^3+ \\ +\vartheta_{n,7}p^2q+\vartheta_{n,8}pq^2+\vartheta_{n,9}q^3,$$

$$q_n(p,q)=\vartheta_{N+n,1}p+\vartheta_{N+n,2}q+\vartheta_{N+n,3}p^2+ \\ +\vartheta_{N+n,4}pq+\vartheta_{N+n,5}q^2+\vartheta_{N+n,6}p^3+ \\ +\vartheta_{N+n,7}p^2q+\vartheta_{N+n,8}pq^2+\vartheta_{N+n,9}q^3, \quad (22)$$

where $n=\overline{1,...,J\text{-}1,J+1,...,N}$ are numbers of degree of freedoms, J is number of the chosen basic generalized coordinate, $p\equiv p_J$, $q\equiv q_J$.

For determination of coefficients of power series (22) we take derivatives of components of vector functions (21):

$$\dot{p}_n(p,q)=\frac{\partial p_n(p,q)}{\partial p}\dot{p}+\frac{\partial p_n(p,q)}{\partial q}\dot{q}, \\ \dot{q}_n(p,q)=\frac{\partial q_n(p,q)}{\partial p}\dot{p}+\frac{\partial q_n(p,q)}{\partial q}\dot{q}, \quad (23)$$

We can express a component of any line of a vector $\{\dot{V}\}=\{\ddot{U}\}$ in a formula (20) as a function of the chosen pair of phase coordinates. Taking into account series (22) we can write this function in the form:

$$\ddot{u}_n(p_J,q_J)=-\sum_{m=1}^{N}G'_{n,m}q_m- \\ -\sum_{m=1}^{N}K'_{n,m}p_m-\sum_{m=1}^{N}\widehat{K}'_{n,m}(pp_\lambda p_\mu)_m. \quad (24)$$

At the same time this function is the left side of the second formula (23). If expression (24) substituted into (23) the following equations are obtained

$$\dot{p}_n(p,q)=q_n(p,q)= \\ =\frac{\partial p_n(p,q)}{\partial p}q+\frac{\partial p_n(p,q)}{\partial q}\ddot{u}_J(p,q), \\ \dot{q}_n(p,q)=\ddot{u}_n(p,q)= \\ =\frac{\partial q_n(p,q)}{\partial p}q+\frac{\partial q_n(p,q)}{\partial q}\ddot{u}_J(p,q). \quad (25)$$

Gathering terms of like powers in p and q we receive system of algebraic equations for definition of coefficients $\vartheta_{n,m}$ [11].

We substitute the calculated coefficients $\vartheta_{n,m}$ into that equation of system (20) which phase coordinates are chosen as p and q. Executing transformations we receive one differential equation of the movement on the chosen vibration mode:

$$\ddot{p}+B_1\dot{p}+B_2p+B_3p^2+B_4p\dot{p}+B_5\dot{p}^2+ \\ +B_6p^3+B_7p^2\dot{p}+B_8p\dot{p}^2+B_9\dot{p}^3=0. \quad (26)$$

The equation (26) is solved by method of harmonious balance.

Results of numerical researches

Rotor parameters are as follows: L = 0.34 m is a shaft length; l = 0.06 m is a length of the console end; d_1 = 0.025 m is a diameter of the console end of a shaft; d_2 = 0.025 m is a diameter of a shaft between supports; $E=2.1\cdot10^{11}$ Pa and $\rho=0.3$; m = 5.0 kg, I_x = 0.1 kg·m^2, I_z = 0.2 kg·m^2; $f_\Omega=\Omega/2\pi$ = 50 Hz is a rotation frequency of a rotor. The standard angular contact ball-bearing parameters are as follows: α = 15° is contact angle; R_2 = 27.525 mm is the radius of outer race; R_1 = 16.000 mm is the radius of inner race; R_K = 5.930 mm is the race radius of curvature; d_B = 11.510 mm is diameter of a ball; N_B = 7 is number of balls; $E=2{,}1\cdot10^{11}$ Pa; μ = 0,3.

Frequencies of transverse oscillations of the linearized system are 70,54 Hz, 110,63 Hz, 196,22 Hz and 202,11 Hz. Frequency of longitudinal oscillations of the linearized system is 101,70 Hz. Backbone curves of a rotor are shown in fig. 2.

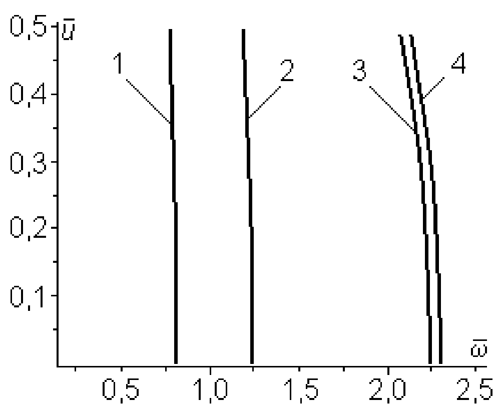

Fig. 2. Backbone curves of a rotor

The system has soft characteristics. Curves 1 and 2 correspond to oscillations in the fundamental mode. With a bigger frequency the curved shaft axle rotates in the direction of shaft rotation, and with a smaller frequency in opposite direction. Curves 3 and 4 correspond to similar oscillations in the second mode.

The fundamental mode of an elastic shaft axle at transverse oscillations when frequency is near 110 Hz is shown in fig. 3 (corresponds to curve 2 in fig. 2).

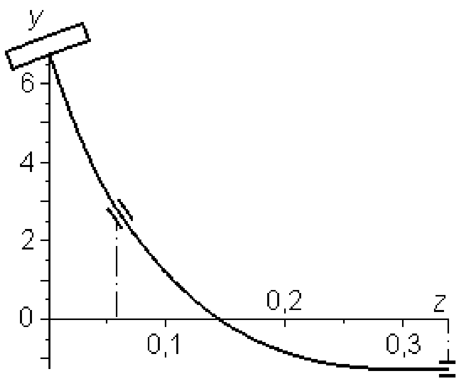

Fig. 3. Fundamental mode of shaft oscillations

At oscillations in the fundamental mode the shaft spindles are from the opposite sides from an axis of bearings. The mode of an elastic shaft axle when frequency is near 202 Hz is shown in fig. 4 (corresponds to curve 4 in fig. 2). In this case the shaft spindles are on the same side from an axis of bearings.

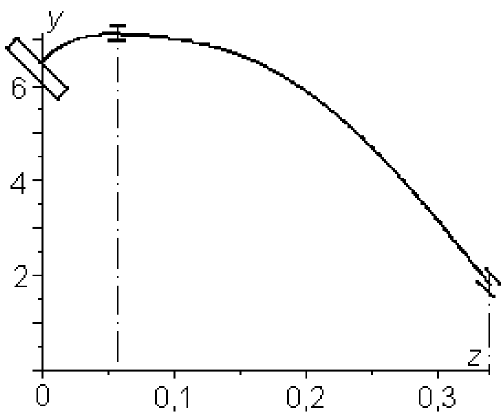

Fig. 4. Second mode of shaft oscillations

Conclusions

Oscillations of a rotor supported by the preloaded angular contact ball bearings are investigated. The disk is fixed on the console end of a shaft. Backbone curves and non-linear normal modes by Shaw and Pierre are obtained. The system has soft characteristics. Resonant oscillations can occur throughout the whole frequency range below the principal resonance frequency.

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